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# Statistical properties of Pauli matrices going through noisy channels

Stéphane Attal and Nadine Guillotin-Plantard \*

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## Abstract

We study the statistical properties of the triplet  $(\sigma_x, \sigma_y, \sigma_z)$  of Pauli matrices going through a sequence of noisy channels, modeled by the repetition of a general, trace-preserving, completely positive map. We show a non-commutative central limit theorem for the distribution of this triplet, which shows up a 3-dimensional Brownian motion in the limit with a non-trivial covariance matrix. We also prove a large deviation principle associated to this convergence, with an explicit rate function depending on the stationary state of the noisy channel.

## 1 Introduction

In quantum information theory one of the most important question is to understand and to control the way a quantum bit is modified when transmitted through a quantum channel. It is well-known that realistic transmission channels are not perfect and that they distort the quantum bit they transmit. This transformation of the quantum state is represented by the action of a completely positive map. These are the so-called noisy channels.

The purpose of this article is to study the action of the repetition of a general completely positive map on basic observables. Physically, this model can be thought of as the sequence of transformations of small identical

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pieces of noisy channels on a qubit. It can also be thought of as a discrete approximation of the more realistic model of a quantum bit going through a semigroup of completely positive maps (a Lindblad semigroup).

As basic observables, we consider the triplet  $(\sigma_x, \sigma_y, \sigma_z)$  of Pauli matrices. Under the repeated action of the completely positive map, they behave as a 3-dimensional quantum random walk. The aim of this article is to study the statistical properties of this quantum random walk.

Indeed, for any initial density matrix  $\rho_{in}$ , we study the statistical properties of the empirical average of the Pauli matrices in the successive states  $\Phi^n(\rho_{in})$ ,  $n \geq 0$  where  $\Phi$  is some completely positive and trace-preserving map describing our quantum channel. Quantum Bernoulli random walks studied by Biane in [1] corresponds to the case where  $\Phi$  is the identity map. Biane [1] proved an invariance principle for this quantum random walk when  $\rho_{in} = \frac{1}{2}I$ .

This article is organized as follows. In section two we describe the physical and mathematical setup. In section three we establish a functional central limit theorem for the empirical average of the quantum random walk associated to the Pauli matrices generalizing Biane's result [1]. This central limit theorem involves a 3-dimensional Brownian motion in the limit, whose covariance matrix is non-trivial and depends explicitly on the stationary state of the noisy channel. In section four, we apply our central limit theorem to some explicit cases, in particular to the King-Ruskai-Szarek-Werner representation of completely positive and trace-preserving maps in  $M_2(\mathbb{C})$ . This allows us to compute the limit Brownian motion for the most well-know quantum channels: the depolarizing channel, the phase-damping channel, the amplitude-damping channel. Finally, in the last section, a large deviation principle for the empirical average is proved.

## 2 Model and notations

Let  $M_2(\mathbb{C})$  be the set of  $2 \times 2$  matrices with complex coefficients. The set of  $2 \times 2$  self-adjoint matrices forms a four dimensional real vector subspace of  $M_2(\mathbb{C})$ . A convenient basis  $\mathcal{B}$  is given by the following matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where  $\sigma_x, \sigma_y, \sigma_z$  are the traditional Pauli matrices, they satisfy the commutation relations:  $[\sigma_x, \sigma_y] = 2i\sigma_z$ , and those obtained by cyclic permutations

of  $\sigma_x, \sigma_y, \sigma_z$ . A state on  $M_2(\mathbb{C})$  is given by a density matrix (i.e. a positive semi-definite matrix with trace one) which we will suppose to be of the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & 1 - \alpha \end{pmatrix}$$

where  $0 \leq \alpha \leq 1$  and  $|\beta|^2 \leq \alpha(1 - \alpha)$ . The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map  $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ .

Let  $M_1, M_2, \dots, M_k, \dots$  be infinitely many copies of  $M_2(\mathbb{C})$ . For each given state  $\rho$ , we consider the algebra

$$\mathcal{M}_\rho = M_1 \otimes M_2 \otimes \dots \otimes M_k \otimes \dots$$

where the product is taken in the sense of  $W^*$ -algebra with respect to the product state

$$\omega = \rho \otimes \Phi(\rho) \otimes \Phi^2(\rho) \otimes \dots \otimes \Phi^k(\rho) \otimes \dots$$

Our main hypothesis is the following. We assume that for any state  $\rho$ , the sequence  $\Phi^n(\rho)$  converges to a stationary state  $\rho_\infty$ , which we write as

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \bar{\beta}_\infty & 1 - \alpha_\infty \end{pmatrix}$$

where  $0 \leq \alpha_\infty \leq 1$  and  $|\beta_\infty|^2 \leq \alpha_\infty(1 - \alpha_\infty)$ .

Put

$$v_1 = 2 \operatorname{Re}(\beta_\infty), v_2 = -2 \operatorname{Im}(\beta_\infty), v_3 = 2\alpha_\infty - 1.$$

For every  $k \geq 1$ , we define

$$x_k = I \otimes \dots \otimes I \otimes (\sigma_x - v_1 I) \otimes I \otimes \dots$$

$$y_k = I \otimes \dots \otimes I \otimes (\sigma_y - v_2 I) \otimes I \otimes \dots$$

$$z_k = I \otimes \dots \otimes I \otimes (\sigma_z - v_3 I) \otimes I \otimes \dots$$

where each  $(\sigma - v I)$  appears on the  $k^{th}$  place.

For every  $n \geq 1$ , put

$$X_n = \sum_{k=1}^n x_k, \quad Y_n = \sum_{k=1}^n y_k, \quad Z_n = \sum_{k=1}^n z_k$$

with initial conditions

$$X_0 = Y_0 = Z_0 = 0.$$

The integer part of a real  $t$  is denoted by  $[t]$ . To each process we associate a continuous time normalized process denoted by

$$X_t^{(n)} = n^{-1/2} X_{[nt]}, \quad Y_t^{(n)} = n^{-1/2} Y_{[nt]}, \quad Z_t^{(n)} = n^{-1/2} Z_{[nt]}.$$

### 3 A central limit theorem

The aim of our article is to study the asymptotical properties of the quantum process  $(X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)})$  when  $n$  goes to infinity. This process being truly non-commutative, there is no hope to obtain an asymptotic behaviour in the classical sense.

For any polynomial  $P = P(X_1, X_2, \dots, X_m)$  of  $m$  variables, we denote by  $\widehat{P}$  the *totally symmetrized polynomial* of  $P$  obtained by symmetrizing each monomial in the following way:

$$X_{i_1} X_{i_2} \dots X_{i_k} \longrightarrow \frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \dots X_{i_{\sigma(k)}}$$

where  $S_k$  is the group of permutations of  $\{1, \dots, k\}$ .

**Theorem 3.1.** *Assume that*

$$(A) \quad \Phi^n(\rho) = \rho_\infty + o\left(\frac{1}{\sqrt{n}}\right).$$

*Then, for any polynomial  $P$  of  $3m$  variables, for any  $(t_1, \dots, t_m)$  such that  $0 \leq t_1 < t_2 < \dots < t_m$ , the following convergence holds:*

$$\begin{aligned} \lim_{n \rightarrow +\infty} w \left[ \widehat{P}(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \right] \\ = \mathbb{E} \left[ P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \dots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)}) \right] \end{aligned}$$

*where  $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$  is a three-dimensional centered Brownian motion with covariance matrix  $Ct$ , where*

$$C = \begin{pmatrix} 1 - v_1^2 & -v_1 v_2 & -v_1 v_3 \\ -v_1 v_2 & 1 - v_2^2 & -v_2 v_3 \\ -v_1 v_3 & -v_2 v_3 & 1 - v_3^2 \end{pmatrix}.$$

**Remark :**

Theorem 3.1 has to be compared with the quantum central limit theorem obtained in [5] and [9]. In our case, the state under which the convergence holds does not need to be an infinite tensor product of states. We also give here a functional version of the central limit theorem. Finally, in [5] (see Remark 3 p.131), the limit is described as a so-called quasi-free state in quantum mechanics. We prove in Theorem 3.1 that the limit is real Gaussian for the class of totally symmetrized polynomials.

*Proof:*

Let  $m \geq 1$  and  $(t_0, t_1, \dots, t_m)$  such that  $t_0 = 0 < t_1 < t_2 < \dots < t_m$ . The polynomial  $P(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)})$  can be rewritten as a polynomial function  $Q$  of the increments:  $X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, X_{t_2}^{(n)} - X_{t_1}^{(n)}, Y_{t_2}^{(n)} - Y_{t_1}^{(n)}, Z_{t_2}^{(n)} - Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)} - X_{t_{m-1}}^{(n)}, Y_{t_m}^{(n)} - Y_{t_{m-1}}^{(n)}, Z_{t_m}^{(n)} - Z_{t_{m-1}}^{(n)}$ . A monomial of  $Q$  is a product of the form  $A_{i_1} \dots A_{i_k}$  for some distinct  $i_1, \dots, i_k$  in  $\{1, \dots, m\}$  where  $A_i$  is a product depending only on the increments  $X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}, Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}, Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}$ . Since the  $A_i$ 's are commuting variables, the totally symmetrized polynomial of the monomial  $A_{i_1} \dots A_{i_k}$  is equal to the product  $\widehat{A_{i_1}} \dots \widehat{A_{i_k}}$ . Consequently, it is enough to prove the theorem for any polynomial  $A_i$ .

Let  $i \geq 1$  fixed, for every  $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ , we begin by determining the asymptotic distribution of the linear combination

$$(\nu_1^2 + \nu_2^2 + \nu_3^2)^{-1/2} \left( \nu_1(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}) + \nu_2(Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}) + \nu_3(Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}) \right) \quad (1)$$

which can be rewritten as

$$\frac{1}{\sqrt{n}} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \left( \frac{\nu_1 x_k + \nu_2 y_k + \nu_3 z_k}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \right).$$

Consider the matrix

$$\begin{aligned} A &= \frac{1}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \left( \nu_1(\sigma_x - v_1 I) + \nu_2(\sigma_y - v_2 I) + \nu_3(\sigma_z - v_3 I) \right) \\ &= \frac{1}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \begin{pmatrix} -\nu_1 v_1 - \nu_2 v_2 + \nu_3(1 - v_3) & \nu_1 - i\nu_2 \\ \nu_1 + i\nu_2 & -\nu_1 v_1 - \nu_2 v_2 - \nu_3(1 + v_3) \end{pmatrix} \end{aligned}$$

which we denote by

$$\begin{pmatrix} a_1 & a_3 \\ \bar{a}_3 & a_2 \end{pmatrix},$$

$a_1, a_2 \in \mathbb{R}, a_3 \in \mathbb{C}$ .

From assumption **(A)** we can write, for every  $n \geq 0$

$$\Phi^n(\rho) = \begin{pmatrix} \alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\ \bar{\beta}_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4) \end{pmatrix}$$

where each sequence  $(\phi_n(i))_n$  satisfies:  $\phi_n(i) = o(1/\sqrt{n})$ .

Let  $k \geq 1$ , the expectation and the variance of  $A$  in the state  $\Phi^k(\rho)$  are respectively equal to

$$\text{Trace}(A\Phi^k(\rho))$$

and

$$\text{Trace}(A^2\Phi^k(\rho)) - \text{Trace}(A\Phi^k(\rho))^2.$$

If both following conditions are satisfied:

$$\sum_{k=[nt_{i-1}]+1}^{[nt_i]} \text{Trace}(A\Phi^k(\rho)) = o(\sqrt{n}) \quad (2)$$

and

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} [\text{Trace}(A^2\Phi^k(\rho)) - \text{Trace}(A\Phi^k(\rho))^2] = a(t_i - t_{i-1}), \quad (3)$$

then (see Theorem 2.8.42 in [3]) the asymptotic distribution of (1) is the Normal distribution  $\mathcal{N}(0, a(t_i - t_{i-1}))$ ,  $a > 0$ .

Let us first prove (2). For every  $k \geq 1$ , a simple computation gives

$$\text{Trace}(A\Phi^k(\rho)) = [a_1\alpha_\infty + a_3\bar{\beta}_\infty + \bar{a}_3\beta_\infty + a_2(1 - \alpha_\infty) + o(1/\sqrt{n})] = o(1/\sqrt{n}),$$

hence

$$\sum_{k=[nt_{i-1}]+1}^{[nt_i]} \text{Trace}(A\Phi^k(\rho)) = \sum_{k=[nt_{i-1}]+1}^{[nt_i]} o(1/\sqrt{n}) = o(\sqrt{n}).$$

This gives (2).

Let us prove (3). Note that the sequence  $(\text{Trace}(A\Phi^n(\rho)))_n$  converges to 0, as  $n$  tends to infinity. As a consequence, it is enough to prove that

$$\frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \text{Trace}(A^2\Phi^k(\rho))$$

converges to a strictly positive constant. A straightforward computation gives

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \text{Trace}(A^2 \Phi^k(\rho)) \\
&= a_1^2 \alpha_\infty + a_2^2 (1 - \alpha_\infty) + |a_3|^2 + (a_1 + a_2)(a_3 \bar{\beta}_\infty + \bar{a}_3 \beta_\infty) \\
&= \frac{(t_i - t_{i-1})}{\nu_1^2 + \nu_2^2 + \nu_3^2} \left[ \nu_1^2 (1 - v_1^2) + \nu_2^2 (1 - v_2^2) + \nu_3^2 (1 - v_3^2) \right. \\
&\quad \left. - 2\nu_1 \nu_2 v_1 v_2 - 2\nu_1 \nu_3 v_1 v_3 - 2\nu_2 \nu_3 v_2 v_3 \right].
\end{aligned}$$

This means that, for every  $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ , for any  $p \geq 1$ , the expectation

$$w \left[ \left( \nu_1 (X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}) + \nu_2 (Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}) + \nu_3 (Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}) \right)^p \right]$$

converges to

$$\mathbb{E} \left[ \left( \nu_1 (B_{t_i}^{(1)} - B_{t_{i-1}}^{(1)}) + \nu_2 (B_{t_i}^{(2)} - B_{t_{i-1}}^{(2)}) + \nu_3 (B_{t_i}^{(3)} - B_{t_{i-1}}^{(3)}) \right)^p \right],$$

where  $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$  is a 3-dimensional Brownian motion with the announced covariance matrix.

The polynomial

$$\left( \nu_1 (X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}) + \nu_2 (Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}) + \nu_3 (Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}) \right)^p$$

can be developed as the sum

$$\sum_{0 \leq p_1 + p_2 \leq p} \nu_1^{p_1} \nu_2^{p_2} \nu_3^{p - p_1 - p_2} \sum_{\mathcal{P}} S_1 S_2 \dots S_p$$

where the summation in the last sum runs over all partitions  $\mathcal{P} = \{A, B, C\}$  of  $\{1, \dots, p\}$  such that  $|A| = p_1, |B| = p_2, |C| = p - p_1 - p_2$ , with the convention:

$$S_j = \begin{cases} X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} & \text{if } j \in A \\ Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)} & \text{if } j \in B \\ Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)} & \text{if } j \in C. \end{cases}$$

The expectation under  $w$  of the above expression converges to the corresponding expression involving the expectation ( $\mathbb{E}[\cdot]$ ) of the Brownian motion  $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$ . As this holds for any  $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ , we deduce that



$w[\sum_{\mathcal{P}} S_1 S_2 \dots S_p]$  converges to the corresponding expectation for the Brownian motion.

We can conclude the proof by noticing that  $\widehat{A}_i$  can be written, modulo multiplication by a constant, as  $\sum_{\mathcal{P}} S_1 S_2 \dots S_p$  for some  $p$ .  $\triangle$

Let us discuss the class of polynomials for which Theorem 3.1 holds. In the particular case when the map  $\Phi$  is the identity map and  $\rho = 1/2I$  (in that case  $v_i = 0$  for  $i = 1, 2, 3$  and  $C = I$ ), Biane [1] proved the convergence of the expectations in Theorem 3.1 for any polynomial in  $3m$  non-commuting variables. It is a natural question to ask whether our result holds for any polynomial  $P$  instead of  $\widehat{P}$ , or at least for a larger class.

Let us give an example of a polynomial for which the convergence in our setting does not hold. Take  $P(X, Y) = XY$ . From Theorem 3.1, the expectation under the state  $\omega$  of

$$X_t^{(n)} Y_t^{(n)} + Y_t^{(n)} X_t^{(n)}$$

converges as  $n \rightarrow +\infty$  to  $2 \mathbb{E}[B_t^{(1)} B_t^{(2)}]$ .

Since we have the following commutation relations

$$[(\sigma_x - v_1 I), (\sigma_y - v_2 I)] = 2i\sigma_z, \quad [(\sigma_y - v_2 I), (\sigma_z - v_3 I)] = 2i\sigma_x$$

and

$$[(\sigma_z - v_3 I), (\sigma_x - v_1 I)] = 2i\sigma_y,$$

we deduce that

$$[X_t^{(n)}, Y_t^{(n)}] = 2in^{-1/2} Z_t^{(n)} + 2itv_3 I, \quad [Y_t^{(n)}, Z_t^{(n)}] = 2in^{-1/2} X_t^{(n)} + 2itv_1 I$$

and

$$[Z_t^{(n)}, X_t^{(n)}] = 2in^{-1/2} Y_t^{(n)} + 2itv_2 I. \quad (4)$$

Then the expectation under the state  $\omega$  of

$$X_t^{(n)} Y_t^{(n)} = \frac{1}{2} \left[ \widehat{P}(X, Y) + [X_t^{(n)}, Y_t^{(n)}] \right]$$

converges to  $\mathbb{E}[B_t^{(1)} B_t^{(2)}] + itv_3 \neq \mathbb{E}[B_t^{(1)} B_t^{(2)}]$ , if  $v_3$  is non zero.

Furthermore, by considering the polynomial  $P(X, Y) = XY^3 + Y^3X$ , it is possible to show that the convergence in Theorem 3.1 can not be enlarged

to the class of symmetric polynomials. Straightforward computations gives that  $P(X, Y)$  can be rewritten as

$$\widehat{XY^3} + \widehat{YX^3} + \frac{3}{4}[X, Y](Y^2 - X^2) + \frac{1}{2}(Y[X, Y]Y - X[X, Y]X) + \frac{1}{4}(Y^2 - X^2)[X, Y]$$

so the expectation  $w[P(X_t^{(n)}, Y_t^{(n)})]$  converges as  $n$  tends to  $+\infty$  to

$$\mathbb{E}[P(B_t^{(1)}, B_t^{(2)})] + 3iv_3t(v_1^2 - v_2^2)$$

which is not equal to  $\mathbb{E}[P(B_t^{(1)}, B_t^{(2)})]$  if  $v_3 \neq 0$  and  $|v_1| \neq |v_2|$ .

In the following corollary we give a condition under which the convergence in Theorem 3.1 holds for any polynomial in  $3m$  non-commuting variables.

**Corollaire 3.1.** *In the case when  $\rho_\infty$  is equal to  $\frac{1}{2}I$ , the convergence holds for any polynomial  $P$  in  $3m$  non-commuting variables, i.e. for every  $t_1 < t_2 < \dots < t_m$ , the following convergence holds:*

$$\begin{aligned} \lim_{n \rightarrow +\infty} w \left[ P(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \right] \\ = \mathbb{E} \left[ P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \dots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)}) \right] \end{aligned}$$

where  $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$  is a three-dimensional centered Brownian motion with covariance matrix  $tI_3$ .

*Proof:*

We consider the polynomials of the form  $S = \frac{1}{N} \sum_{\mathcal{P}} S_1 S_2 \dots S_{p_1+p_2+p_3}$  where the summation is done over all partitions  $\mathcal{P} = \{A, B, C\}$  of the set  $\{1, \dots, p_1+p_2+p_3\}$  such that  $|A| = p_1, |B| = p_2, |C| = p_3$ , with the convention:

$$S_j = \begin{cases} X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} & \text{if } j \in A \\ Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)} & \text{if } j \in B \\ Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)} & \text{if } j \in C \end{cases}$$

and  $N$  is the number of terms in the sum.

From Theorem 3.1 the expectation under the state  $w$  of  $S$  converges to

$$\mathbb{E} \left[ \prod_{j=1}^3 (B_{t_i}^{(j)} - B_{t_{i-1}}^{(j)})^{p_j} \right].$$

Using the commutation relations (4) with the  $v_i$ 's being all equal to zero, monomials of  $S$  differ of each other by  $n^{-1/2}$  times a polynomial of total degree less or equal to  $(p_1 + p_2 + p_3) - 1$ . It is easy to conclude by induction.  $\triangle$

## 4 Examples

### 4.1 King-Ruskai-Szarek-Werner's representation

The set of  $2 \times 2$  self-adjoint matrices forms a four dimensional real vector subspace of  $M_2(\mathbb{C})$ . A convenient basis of this space is given by  $\mathcal{B} = \{I, \sigma_x, \sigma_y, \sigma_z\}$ . Each state  $\rho$  on  $M_2(\mathbb{C})$  can then be written as

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

where  $x, y, z$  are reals such that  $x^2 + y^2 + z^2 \leq 1$ . Equivalently, in the basis  $\mathcal{B}$ ,

$$\rho = \frac{1}{2}(I + x \sigma_x + y \sigma_y + z \sigma_z)$$

with  $x, y, z$  defined above. Thus, the set of density matrices can be identified with the unit ball in  $\mathbb{R}^3$ . The pure states, that is, the ones for which  $x^2 + y^2 + z^2 = 1$ , constitutes the Bloch sphere.

The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map  $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ . Kraus and Choi [2, 7, 8] gave an abstract representation of these particular maps in terms of Kraus operators: There exists at most four matrices  $L_i$  such that for any density matrix  $\rho$ ,

$$\Phi(\rho) = \sum_{1 \leq i \leq 4} L_i^* \rho L_i$$

with  $\sum_i L_i L_i^* = I$ . The matrices  $L_i$  are usually called the *Kraus operators* of  $\Phi$ . This representation is unique up to a unitary transformation. Recently, King, Ruskai et al [10, 6] obtained a precise characterization of completely positive and trace-preserving maps from  $M_2(\mathbb{C})$  as follows. The map  $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  being linear and preserving the trace, it can be represented as a unique  $4 \times 4$ -matrix in the basis  $\mathcal{B}$  given by

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix}$$

with  $\mathbf{0} = (0, 0, 0)$ ,  $\mathbf{t} \in \mathbb{R}^3$  and  $\mathbf{T}$  a real  $3 \times 3$ -matrix. King, Ruskai et al [10, 6]

proved that via changes of basis, this matrix can be reduced to

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} \quad (5)$$

Necessary and sufficient conditions under which the map  $\Phi$  with reduced matrix  $T$  for which  $|t_3| + |\lambda_3| \leq 1$  is completely positive are (see [6])

$$(\lambda_1 + \lambda_2)^2 \leq (1 + \lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2) \left( \frac{1 + \lambda_3 \pm t_3}{1 - \lambda_3 \pm t_3} \right) \leq (1 + \lambda_3)^2 - t_3^2 \quad (6)$$

$$(\lambda_1 - \lambda_2)^2 \leq (1 - \lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2) \left( \frac{1 - \lambda_3 \pm t_3}{1 + \lambda_3 \pm t_3} \right) \leq (1 - \lambda_3)^2 - t_3^2 \quad (7)$$

$$\begin{aligned} & [1 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (t_1^2 + t_2^2 + t_3^2)]^2 \\ & \geq 4 [\lambda_1^2(t_1^2 + \lambda_2^2) + \lambda_2^2(t_2^2 + \lambda_3^2) + \lambda_3^2(t_3^2 + \lambda_1^2) - 2\lambda_1\lambda_2\lambda_3] . \end{aligned} \quad (8)$$

We now apply Theorem 3.1 in this setting. Let  $\Phi$  be a completely positive and trace preserving map with matrix  $T$  given in (5), with coefficients  $t_i, \lambda_i, i = 1, 2, 3$  satisfying conditions (6), (7) and (8). Moreover, we assume that  $|\lambda_i| < 1, i = 1, 2, 3$ . For every  $n \geq 0$ ,

$$\Phi^n(\rho) = \frac{1}{2} \begin{pmatrix} 1 + \phi_n(3) & \phi_n(1) - i \phi_n(2) \\ \phi_n(1) + i \phi_n(2) & 1 - \phi_n(3) \end{pmatrix}$$

where the sequences  $(\phi_n(i))_{n \geq 0}, i = 1, 2, 3$  satisfy the induction relations:

$$\phi_n(i) = \lambda_i \phi_{n-1}(i) + t_i.$$

with initial conditions  $\phi_0(1) = x, \phi_0(2) = y$  and  $\phi_0(3) = z$ . Explicit formulae can easily be obtained. We get, for every  $n \geq 0$ ,

$$\phi_n(1) = \left( x - \frac{t_1}{1 - \lambda_1} \right) \lambda_1^n + \frac{t_1}{1 - \lambda_1}$$

$$\phi_n(2) = \left( y - \frac{t_2}{1 - \lambda_2} \right) \lambda_2^n + \frac{t_2}{1 - \lambda_2}$$

$$\phi_n(3) = \left( z - \frac{t_3}{1 - \lambda_3} \right) \lambda_3^n + \frac{t_3}{1 - \lambda_3}.$$

Hence, for any state  $\rho$ , for any  $n \geq 1$ ,

$$\Phi_n(\rho) = \rho_\infty + o(|\lambda|_{max}^n)$$

where  $|\lambda|_{max} = \max_{i=1,2,3} |\lambda_i|$  and

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \bar{\beta}_\infty & 1 - \alpha_\infty \end{pmatrix}$$

with  $\alpha_\infty = \frac{1}{2} \left( 1 + \frac{t_3}{1 - \lambda_3} \right)$  and  $\beta_\infty = \frac{1}{2} \left( \frac{t_1}{1 - \lambda_1} - i \frac{t_2}{1 - \lambda_2} \right)$ . Theorem 3.1 applies with  $v_i = \frac{t_i}{1 - \lambda_i}, i = 1, 2, 3$ .

We now give some examples of well-known quantum channels. For each of them we give their Kraus operators, their corresponding matrix  $T$  in the King-Ruskai-Szarek-Werner's representation, as well as the vector  $v = (v_1, v_2, v_3)$  and the covariance matrix  $C$  obtained in Theorem 3.1. It is worth noticing that if  $\Phi$  is a *unital* map, i.e. such that  $\Phi(I) = I$ , then the covariance matrix  $C$  is equal to the identity matrix  $I_3$ .

1. *The depolarizing channel:*

Kraus operators: for some  $0 \leq p \leq 1$ ,

$$L_1 = \sqrt{1-p}I, L_2 = \sqrt{\frac{p}{3}}\sigma_x, L_3 = \sqrt{\frac{p}{3}}\sigma_y, L_4 = \sqrt{\frac{p}{3}}\sigma_z.$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{4p}{3} & 0 & 0 \\ 0 & 0 & 1 - \frac{4p}{3} & 0 \\ 0 & 0 & 0 & 1 - \frac{4p}{3} \end{pmatrix}$$

The vector  $v$  is the null vector and the covariance matrix  $C$  in this case is given by the identity matrix  $I_3$ .

2. *Phase-damping channel:*

Kraus operators: for some  $0 \leq p \leq 1$ ,

$$L_1 = \sqrt{1-p} I, \quad L_2 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad L_3 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The vector  $v$  is the null vector and the covariance matrix  $C$  in this case is given by  $I_3$ .

3. *Amplitude-damping channel:*

Kraus operators: for some  $0 \leq p \leq 1$ ,

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ t & 0 & 0 & 1-p \end{pmatrix}$$

The vector  $v$  is equal to  $(0, 0, 1)$ . The covariance matrix in this case is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. *Trigonometric parameterization:*

Consider the particular Kraus operators

$$L_1 = \left[ \cos\left(\frac{v}{2}\right) \cos\left(\frac{u}{2}\right) \right] I + \left[ \sin\left(\frac{v}{2}\right) \sin\left(\frac{u}{2}\right) \right] \sigma_z$$

and

$$L_2 = \left[ \sin\left(\frac{v}{2}\right) \cos\left(\frac{u}{2}\right) \right] \sigma_x - i \left[ \cos\left(\frac{v}{2}\right) \sin\left(\frac{u}{2}\right) \right] \sigma_y.$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix}$$

The vector  $v$  is equal to  $(0, 0, \frac{\sin u \sin v}{1 - \cos u \cos v})$ . The covariance matrix in this case is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - v_3^2 \end{pmatrix}$$

with  $v_3 = \frac{\sin u \sin v}{1 - \cos u \cos v}$ .

## 4.2 CP map associated to a Markov chain

With every Markov chain with two states and transition matrix given by

$$P = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}, \quad p, q \in (0, 1)$$

is associated a completely positive and trace preserving map, denoted by  $\Phi$ , with the Kraus operators:

$$L_1 = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} = \frac{\sqrt{p}}{2}(I + \sigma_z) + \frac{\sqrt{1-p}}{2}(\sigma_x + i\sigma_y)$$

and

$$L_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{q} & \sqrt{1-q} \end{pmatrix} = \frac{\sqrt{1-q}}{2}(I - \sigma_z) + \frac{\sqrt{q}}{2}(\sigma_x - i\sigma_y).$$

Let  $\rho$  be the density matrix

$$\frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

where  $x, y, z$  are reals such that  $x^2 + y^2 + z^2 \leq 1$ . The map  $\Phi$  transforms the density matrix  $\rho$  into a new one given by

$$\Phi(\rho) = L_1^* \rho L_1 + L_2^* \rho L_2.$$

By induction, for every  $n \geq 0$ ,

$$\Phi^n(\rho) = \begin{pmatrix} p_n & r_n \\ r_n & 1 - p_n \end{pmatrix}$$

where the sequences  $(p_n)_{n \geq 0}$ , and  $(r_n)_{n \geq 0}$  satisfy the recurrence relations: for every  $n \geq 1$ ,

$$p_n = p_{n-1}(p - q) + q$$

and

$$r_n = \sqrt{q(1 - q)} + p_{n-1}(\sqrt{p(1 - p)} - \sqrt{q(1 - q)})$$

with the initial condition  $p_0 = (1 + z)/2$ . Assumption (A) is then clearly satisfied with

$$\rho_\infty = \frac{1}{1 + q - p} \begin{pmatrix} q & \beta \\ \beta & 1 - p \end{pmatrix}$$

where  $\beta = [q\sqrt{p(1 - p)} + (1 - p)\sqrt{q(1 - q)}]$ . Then, applying Theorem 3.1, if  $P$  is a polynomial of  $3m$  non-commuting variables, for every  $0 < t_1 < t_2 < \dots < t_m$ , the following convergence holds

$$\begin{aligned} \lim_{n \rightarrow +\infty} w \left[ \widehat{P}(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \right] \\ = \mathbb{E} \left[ P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \dots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)}) \right] \end{aligned}$$

where  $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$  is a three-dimensional centered Brownian motion with Covariance matrix  $Ct$  where

$$C = \begin{pmatrix} 1 - v_1^2 & 0 & -v_1 v_2 \\ 0 & 1 & 0 \\ -v_1 v_2 & 0 & 1 - v_2^2 \end{pmatrix}$$

with

$$v_1 = \frac{2}{1 + q - p} [q\sqrt{p(1 - p)} + (1 - p)\sqrt{q(1 - q)}]$$

and

$$v_2 = \frac{p + q - 1}{1 + q - p}.$$



## 5 Large deviation principle

Let  $\Gamma$  be a Polish space endowed with the Borel  $\sigma$ -field  $\mathcal{B}(\Gamma)$ . A good *rate function* is a lower semi-continuous function  $\Lambda^* : \Gamma \rightarrow [0, \infty]$  with compact level sets  $\{x; \Lambda^*(x) \leq \alpha\}, \alpha \in [0, \infty[$ . Let  $v = (v_n)_n \uparrow \infty$  be an increasing sequence of positive reals. A sequence of random variables  $(Y_n)_n$  with values in  $\Gamma$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is said to satisfy a *Large Deviation Principle* (LDP) with speed  $v = (v_n)_n$  and the good rate function  $\Lambda^*$  if for every Borel set  $B \in \mathcal{B}(\Gamma)$ ,

$$\begin{aligned} - \inf_{x \in B^o} \Lambda^*(x) &\leq \liminf_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \\ &\leq \limsup_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq - \inf_{x \in \bar{B}} \Lambda^*(x). \end{aligned}$$

For every  $k \geq 1$ , we define

$$\begin{aligned} \bar{x}_k &= I \otimes \dots \otimes I \otimes \sigma_x \otimes I \otimes \dots \\ \bar{y}_k &= I \otimes \dots \otimes I \otimes \sigma_y \otimes I \otimes \dots \\ \bar{z}_k &= I \otimes \dots \otimes I \otimes \sigma_z \otimes I \otimes \dots \end{aligned}$$

where each  $\sigma_i$  appears on the  $k^{th}$  place.

For every  $n \geq 1$ , we consider the processes

$$\bar{X}_n = \sum_{k=1}^n \bar{x}_k, \quad \bar{Y}_n = \sum_{k=1}^n \bar{y}_k, \quad \bar{Z}_n = \sum_{k=1}^n \bar{z}_k$$

with initial conditions

$$\bar{X}_0 = \bar{Y}_0 = \bar{Z}_0 = 0.$$

To each vector  $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$ , we associate the euclidean norm  $\|\nu\| = \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}$  and  $\langle \cdot, \cdot \rangle$  the corresponding inner product.

**Theorem 5.1.** *Let  $\Phi$  be a completely positive and trace-preserving map for which there exists a state*

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \bar{\beta}_\infty & 1 - \alpha_\infty \end{pmatrix}$$

*such that for any given state  $\rho$ ,*

$$\Phi^n(\rho) = \rho_\infty + o(1).$$

For every  $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^{3,*}$ , the sequence

$$\left( \frac{\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n}{n} \right)_{n \geq 1}$$

satisfies a LDP with speed  $n$  and the good rate function

$$I(x) = \begin{cases} \frac{1}{2} \left[ \left( 1 + \frac{x}{\|\nu\|} \right) \log \left( \frac{\|\nu\| + x}{\|\nu\| + \langle \nu, v \rangle} \right) \right. \\ \quad \left. + \left( 1 - \frac{x}{\|\nu\|} \right) \log \left( \frac{\|\nu\| - x}{\|\nu\| - \langle \nu, v \rangle} \right) \right] & \text{if } |x| < \|\nu\|. \\ +\infty & \text{otherwise.} \end{cases}$$

where  $v_1 = 2 \operatorname{Re}(\beta_\infty)$ ,  $v_2 = -2 \operatorname{Im}(\beta_\infty)$ ,  $v_3 = 2\alpha_\infty - 1$ .

*Proof:*

The matrix

$$B := \nu_1 \sigma_x + \nu_2 \sigma_y + \nu_3 \sigma_z = \begin{pmatrix} \nu_3 & \nu_1 - i\nu_2 \\ \nu_1 + i\nu_2 & -\nu_3 \end{pmatrix}$$

has two distinct eigenvalues  $\pm \|\nu\|$ .

For every  $n \geq 0$ , we can write

$$\Phi^n(\rho) = \begin{pmatrix} \alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\ \bar{\beta}_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4) \end{pmatrix}$$

where the four sequences  $(\phi_n(i))_{n \geq 0}$  satisfy  $\phi_n(i) = o(1)$ .

For any  $k \geq 1$ , the expectation of  $B$  in the state  $\Phi^k(\rho)$  is equal to

$$\operatorname{Trace}(B \Phi^k(\rho)) = \langle \nu, v \rangle + \varepsilon_k,$$

with  $\varepsilon_n = o(1)$ . As a consequence, the distribution of  $B$  is

$$p_k(\|\nu\|) = \frac{1}{2} \left[ 1 + \frac{1}{\|\nu\|} (\langle \nu, v \rangle + \varepsilon_k) \right] = 1 - p_k(-\|\nu\|).$$

Using the fact that  $\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n$  is the sum of  $n$  commuting matrices, we get that

$$\begin{aligned} & \frac{1}{n} \log w(\exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n)) \\ &= \frac{1}{n} \sum_{k=1}^n \log \left( e^{\|\nu\|t} p_k(\|\nu\|) + e^{-\|\nu\|t} (1 - p_k(\|\nu\|)) \right) \end{aligned}$$

Since  $\varepsilon_n = o(1)$ , we obtain that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{n} \log w \left( \exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n) \right) \\ &= \log \left( \cosh (||\nu||t) + \frac{\langle \nu, v \rangle}{||\nu||} \sinh (||\nu||t) \right) \\ &= \log \left( \cosh (||\nu||t) \right) + \log \left( 1 + \frac{\langle \nu, v \rangle}{||\nu||} \tanh (||\nu||t) \right). \end{aligned}$$

We denote by  $\Lambda(t)$  this function of  $t$ .

For every  $t \in \mathbb{R}$ , the function  $\Lambda$  is finite and differentiable on  $\mathbb{R}$ , then, by Gärtner-Ellis Theorem (see [4]), the LDP holds with the good rate function

$$I(x) = \sup_{t \in \mathbb{R}} \{tx - \Lambda(t)\}.$$

A simple computation leads to the rate function given in the theorem.

△

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